**Electromagnetic Waves in the Vacuum with Torsion and Spin.**

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Exact radiative wave solutions to the classical homogeneous Maxwell equations in the vacuum have been found that are not transverse, exhibit both torsion and spin, and for which the second Poincare invariant $\mathbf{E} \circ \mathbf{B} \neq 0$. Two four component rank 3 tensors of spin current and torsion are constructed on topological grounds. The divergence of each pseudo vector generates the Poincare invariants of the electromagnetic system.

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## 0. Introduction

In section 1 the domain of classical electromagnetism is defined in terms of four vector fields $\mathbf{D}, \mathbf{E}, \mathbf{B}, \mathbf{H}$, and the vector and scalar potentials $\{\mathbf{A}, \phi\}$. In section 2 prior attempts to find time dependent wave solutions with non-zero Poincare invariants are discussed briefly, with special emphasis placed upon Ranada's use of the Hopf map. In section 3, several time dependent closed form solutions are presented that have $\mathbf{E} \circ \mathbf{B} \neq 0$.

## 1. The Domain of Classical Electromagnetism

In terms of the notation and the language of Sommerfeld and Stratton [1], the classic definition of an electromagnetic system is a domain of space-time independent variables, $\{x, y, z, t\}$, which supports both the Maxwell-Faraday equations,

$$curl\ \mathbf{E} + \partial \mathbf{B}/\partial t = 0, \qquad div\ \mathbf{B} = 0, \tag{1.1}$$

and the Maxwell-Ampere equations,

$$curl\ \mathbf{H} - \partial \mathbf{D}/\partial t = \mathbf{J}, \qquad div\ \mathbf{D} = \rho. \tag{1.2}$$

For the Lorentz vacuum state, the charge-current densities are subsumed to be zero $[\mathbf{J}, \rho] = 0$ and the field excitations, $\mathbf{D}$ and $\mathbf{H}$, are linearly connected to the field intensities, $\mathbf{E}$ and $\mathbf{B}$, by means of the homogeneous and isotropic constitutive relations $\mathbf{D} = \varepsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$. It is further subsumed that the classic Maxwell electromagnetic system is constrained by the statement that the field intensities are deducible from a system of twice differentiable potentials, $[\mathbf{A}, \phi]$:

$$\mathbf{B} = curl\ \mathbf{A}, \quad \mathbf{E} = -grad\ \phi - \partial\mathbf{A}/\partial t. \tag{1.3}$$

This constraint topologically implies that domains that support non-zero values for the covariant field intensities, $\mathbf{E}$ and $\mathbf{B}$, can *not* be compact domains without a boundary, unless the domain has Euler characteristic zero. The only two exceptions are therefore the Torus and the Klein bottle.

Besides the charge current 4-vector density, $[\mathbf{J}, \rho]$, whose integral over any closed 3 dimensional manifold is a deformation invariant of the Maxwell system, there exist two other algebraic combinations of the fields and potentials that can lead to similar topological quantities. These objects are the rank 3 Spin (pseudo) vector, or current [2], defined in component form as

$$\mathbf{S}_4 = [\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi, \mathbf{A} \circ \mathbf{D}] \equiv [\mathbf{S}, \sigma], \tag{1.4}$$

and the rank 3 Torsion (pseudo) vector [3] defined in component form as

$$\mathbf{T}_4 = [\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}] \equiv [\mathbf{T}, h]. \tag{1.5}$$

Note that the classical helicity, $h = \mathbf{A} \circ \mathbf{B}$, forms only the fourth component of this third rank tensor. The derivation of these 4-component tensor fields of rank 3 and their topological implications are developed in more detail elsewhere.[4] The 4-divergence of these 4-component vectors leads to the Poincare projective invariants of the Maxwell system:

$$\begin{aligned} Poincare\, Invariant\ 1 &= div_3(\mathbf{A} \times \mathbf{H} + \mathbf{D}\phi) + \partial(\mathbf{A} \circ \mathbf{D})/\partial t \\ &= (\mathbf{B} \circ \mathbf{H} - \mathbf{D} \circ \mathbf{E}) - (\mathbf{A} \circ \mathbf{J} - \rho\phi) \end{aligned} \tag{1.6}$$

$$\begin{aligned} Poincare\ Invariant\ 2 &= div_3(\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi) + \partial(\mathbf{A} \circ \mathbf{B})/\partial t \\ &= -2\mathbf{E} \circ \mathbf{B} \end{aligned} \tag{1.7}$$

When the Spin vector is non-zero, and its 4-divergence (the first Poincare invariant) vanishes, integrals over closed three manifolds of the Spin 4 vector lead a topological property equivalent to a deRham period integral [5]:

$$Spin = \iiint_{closed} \{S^x dy\hat{}dz\hat{}dt - S^y dx\hat{}dz\hat{}dt + S^z dx\hat{}dy\hat{}dt - \sigma dx\hat{}dy\hat{}dz\}. \tag{1.8}$$

This closed integral is a deformation invariant of any evolutionary process that can be described by a singly parameterized vector field, $\beta\mathbf{V}$, independent of the choice of parameterization, $\beta$, for the Lie derivative of the Spin integral vanishes:

$$L_{(\beta\mathbf{V})}\, Spin = 0. \tag{1.9}$$

When the associated Poincare invariant vanishes, the values of the Spin integral form rational ratios. Similar statements hold for the closed integrals of the Torsion vector.

## 2. Earlier Work

In earlier articles, Chu and Ohkawa [6] developed a standing wave example that led Khare and Pradhan [7] to construct a free space electromagnetic wave which had non-zero Poincare Invariants. Braunstein [8] mentioned that these developments were technically flawed and further argued that the existence of a bonafide (spatially bounded) electromagnetic wave in free space with non-zero Poincare invariants was impossible.

The solution counter examples to Braunstein's claim, as given in section 3 below, were inspired by the work of Ranada [9] who investigated the applications of the Hopf map to the problem of finding knotted solutions to the Maxwell equations. Recall that the Hopf map can be written as the common constraint on the map $\Phi$ from $R4(x,y,z,s)$ to $R3(X,Y,Z)$ given by the expressions:

$$[X,Y,Z] = [2(ys - xz)\,, -2(yz + xs)\ , -(z^2 + s^2) + (x^2 + y^2)] \tag{2.1}$$

From another point of view, the Hopf map defines a family of cyclides in $\{x,y,z\}$ parameterized by $s$

$$Hopf \quad Map\,Cyclide \qquad r^4 + (2s^2 - 1)r^2 + s^4 = 0, \tag{2.2}$$

and where $r^2 = x^2 + y^2 + z^2$. A picture of the Hopf cyclide can be seen in reference [10].

Ranada suggested the 4-potential (based on the Hopf map for $s = 1$)

$$\mathbf{A} = [y, -x, -1](2/\pi)/\lambda^4\ , \quad \phi = 0/\lambda^4, \quad where\ \lambda^2 = 1 + x^2 + y^2 + z^2, \tag{2.3}$$

which will generate the fields

$$\mathbf{E} = [0,0,0] \qquad \mathbf{B} = [-2(y + zx), +2(x - yz), +(-1 + x^2 + y^2 - z^2)](4/\pi)/\lambda^6. \tag{2.4}$$

Note that the components of the induced $\mathbf{B}$ field are precisely the coefficients of the Hopf Map (to within a factor). Ranada discusses the knottedness of the magnetic field lines of such solutions to the Maxwell-Faraday equations, which have finite helicity, but zero second Poincare invariant.

$$h = \mathbf{A} \circ \mathbf{B} = -8s/\pi^2\lambda^8 \ , \qquad \mathbf{E} \circ \mathbf{B} = 0\,. \tag{2.5}$$

Unfortunately, the Ranada 4-potential does not satisfy the Maxwell-Ampere equation for the vacuum with a zero charge current 4-vector, and therefore is not a suitable vacuum solution.

Consider a modification of the Hopf map by substituting $s \Rightarrow ict$ to yield the modified time dependent potentials:

$$\mathbf{A} = [y, -x, +ict](2/\pi)/\lambda^4 \ , \quad \phi = icz/\lambda^4, \quad where\ \lambda^2 = -(ct)^2 + x^2 + y^2 + z^2. \tag{2.6}$$

Such potentials lead to complex $\mathbf{E}$ and $\mathbf{B}$ fields that indeed satisfy (subject to the phase condition $\varepsilon\mu c^2 = 1$) the zero charge current criteria for a vacuum solution, and the vector wave equation. However, the second Poincare invariant is imaginary and the Poynting vector vanishes for such solutions. The Spin vector, on the other hand, is real and has non-zero divergence.

### 3. Example Radiative Vacuum Solutions for which $\mathbf{E} \circ \mathbf{B} \neq 0$.

The modifications of the Hopf map further suggest consideration of the system of potentials given by the equations

$$\mathbf{A} = [+y, -x, -ct]/\lambda^4 \ , \quad \phi = cz/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2. \tag{3.1}$$

which yield the real field intensities,

$$\mathbf{E} = [-2(cty - xz), +2(ctx + yz), -(c^2t^2 + x^2 + y^2 - z^2)]2c/\lambda^6 \tag{3.2}$$

and

$$\mathbf{B} = [-2(cty + xz), +2(ctx - yz), +(c^2t^2 + x^2 + y^2 - z^2)]2/\lambda^6. \tag{3.3}$$

Subject to the dispersion relation, $\varepsilon\mu c^2 = 1$ and the Lorentz constitutive conditions, these time dependent wave functions satisfy the homogeneous Maxwell equations without charge currents, and are therefore acceptable vacuum solutions. The extensive algebra involved in these and other computations in this article were checked with a Maple symbolic mathematics program [11].

The Spin current density for this first non-transverse wave example is evaluated as:

$$\mathbf{S}_4 = [x(3\lambda^2 - 4y^2 - 4x^2), y(3\lambda^2 - 4y^2 - 4x^2), z(\lambda^2 - 4y^2 - 4x^2), t(\lambda^2 - 4y^2 - 4x^2)](2/\mu)/\lambda^{10}, \tag{3.4}$$

and has zero divergence. The Torsion current may be evaluated as

$$\mathbf{T}_4 = -[x, y, z, t]2c/\lambda^8. \tag{3.5}$$

and has a non-zero divergence equal to the second Poincare invariant

$$Poincare\ 2 = -2\mathbf{E} \circ \mathbf{B} = +8c/\lambda^8. \tag{3.6}$$

As the first Poincare invariant is zero it is possible to construct a deformation invariant in terms of the deRham period integral of the Spin current 4 vector over a closed 3 dimensional submanifold.

It is to be noted that the example solution given above is but one of a class of vacuum wave solutions that have similar non transverse properties. As a second example, consider the fields that can be constructed from the potentials,

$$\mathbf{A} = [+ct, -z, +y]/\lambda^4\ , \quad \phi = cx/\lambda^4, \quad where\ \lambda^2 = -c^2t^2 + x^2 + y^2 + z^2. \tag{3.7}$$

These potentials will generate the field intensities

$$\mathbf{E} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(ctz + yx), -2(cty - zx)]2c/\lambda^6 \tag{3.8}$$

and

$$\mathbf{B} = [+(-c^2t^2 + x^2 - y^2 - z^2), +2(-ctz + yx), +2(cty + zx)]2/\lambda^6. \tag{3.9}$$

As before, these fields satisfy the Maxwell-Faraday equations, and the associated excitations satisfy the Maxwell-Ampere equations without producing a charge current 4-vector. However, it follows by direct computation that the second Poincare invariant, and the Torsion 4-vector are of opposite signs to the values computed for the first example:

$$\mathbf{T}_4 = +[x, y, z, t]2c/\lambda^8\,, \qquad -2\mathbf{E} \circ \mathbf{B} = -8c/\lambda^8\ . \tag{3.10}$$

When the two examples are combined by addition (or subtraction), the resulting wave is transverse magnetic (in the topological sense that $\mathbf{A} \circ \mathbf{B} = 0$). Not only does the second Poincare invariant vanish under superposition, but so also does the Torsion 4 vector. Conversely, the examples above show that there can exist transverse magnetic waves which can be decomposed into two non-transverse waves. A notable feature of the superposed solutions is that the Spin 4 vector current does not vanish, hence the example superposition is a wave that is not transverse electric ($\mathbf{A} \circ \mathbf{D} \neq 0$). For the examples presented above and their superposition, the first Poincare invariant vanishes, which implies that the Spin integral remains a conserved topological quantity for the

superposition, with values proportional to the integers. The Spin current density for the combined examples is given by the formula:

$$\begin{aligned}\mathbf{S}_4 &= [-2x(y+ct)^2, (y+ct)(x^2-y^2+z^2-2cty-c^2t^2), -2z(y+ct)^2, \\ &\quad -(y+ct)(x^2+y^2+z^2+2cty+c^2t^2)](4/\mu)/\lambda^{10},\end{aligned} \tag{3.11}$$

while the Torsion current is a zero vector

$$\mathbf{T}_4 = [0,0,0,0]. \tag{3.12}$$

In addition, for the superposed example, the spatial components of the Poynting vector are equal to the Spin current density vector multiplied by $\gamma$, such that

$$\mathbf{E}\times\mathbf{H} = \gamma\ \mathbf{S}, \quad \mathit{with}\ \gamma = -(x^2+y^2+z^2+2cty+c^2t^2)/2c(y+ct)\lambda^2. \tag{3.13}$$

These results seem to give classical credence to the Planck assumption that vacuum state of Maxwell's electrodynamics supports quantized angular momentum, and that the energy flux must come in multiples of the spin quanta. In other words, these combined solutions to classical electrodynamics have some of the qualities of the photon.

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11. A Maple symbolic mathematics program to compute the functions in this article may be found at

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